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On wakes in stratified fluids

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The general nature of the flow at large distances from a two-dimensional body moving uniformly through an unbounded, linearly stratified, non-diffusive viscous fluid is considered. The governing equations are linearized using the Oseen and Boussinesq approximations, and the boundary conditions at the body are replaced by a linearized momentum-integral equation. The solution of this linear problem shows a system of jets upstream and a pattern of waves downstream of the body. The effects of viscosity on these lee waves are considered in detail.

1. Introduction

The flow induced by bodies moving through viscous fluids of uniform density has been studied, with varying degrees of success, for over a century. In 1851, Stokes published his solution for the slow motion of a sphere in which he neglected inertial forces. This solution was not self-consistent in that the neglected inertial forces, as computed from the solution, become as large as the retained viscous forces far from the sphere. Oseen (1910) later proposed a linear approximation for the inertial forces, reworked the problem of the sphere, and found a wake behind the body. A year later, Lamb solved the problem of flow past a circular cylinder. In the early thirties, Tollmien (1931) and Goldstein (1933) obtained similarity solutions valid far downstream of bodies moving through viscous fluids of uniform density and, through momentum considerations, related their solutions to the drag. More recently, the papers of Lagerstrom & Cole (1955), Kaplun (1957), Kaplun & Lagerstrom (1957), and Proudman & Pearson (1957) have pioneered the technique of matching inner and outer expansions of solutions of the Navier–Stokes equations. Using this method, several papers have been published (Chang 1961; Cox 1965; Shi 1965) concerning flows about variously shaped bodies.

In contrast with the great effort that has been put forth in the case of homogeneous flows, little appears to have been expended, thus far, in the case of the motion of bodies through stratified viscous fluids. In 1959, neglecting inertial forces and making the boundary-layer approximation, Long obtained a similarity solution valid far upstream of a body moving with constant speed through a linearly stratified, non-diffusive viscous fluid; he was unable to obtain a downstream solution under the above approximations. When he later considered the diffusive case, Long (1962) found a similarity solution having symmetric streamlines far upstream and downstream of the body. Martin (1966) considered the slow flow of a stratified viscous fluid over a flat plate and has found a similarity solution. Recently, Graebel (1967) considered the slow motion of a body through a strongly stratified, non-diffusive viscous fluid and found a solution valid far upstream of the body and a solution valid near the surface of the body on its downstream side; his solutions neglect the inertia of the fluid. His upstream solution tends to Long's (1959) asymptotic solution and his downstream solution has a damped wave-like nature, with an attentuation constant and a wavelength agreeing with those presented here for the special case of zero Reynolds number. In a paper that has just been published, Bretherton (1967) considers the initial-value problem of a cylinder moving in an inviscid stratified fluid and examines, in detail, the internal gravity waves which produce the flow. His solution does not explain the steady-state existence of waves downstream of the body.

This paper considers the gėneral nature of the flow far upstream and downstream of a body moving through a stratified, non-diffusive viscous fluid. Long's (1959) solution is found to be valid asymptotically far upstream of the body, and this solution is related to the drag of the body; waves, damped by viscosity, are found in the lee of the body. As the speed of the body increases, these lee waves increase in wavelength and decay more slowly.

The method of solution developed in the following sections may be easily extended to include the effects of diffusion. Very far from the body, where the non-diffusive perturbations have become very small, the effects of diffusion become relatively important and Long's (1962) similarity solution becomes valid. As some details of the diffusive solution remain unclear at this time, we shall not discuss it further, and the interested reader is referred to Janowitz (1967).

2. Formulation of the problem

We consider a two-dimensional body moving horizontally to the left, with constant speed U, through an unbounded, linearly stratified, non-diffusive viscous fluid. The body is assumed to experience a drag force per unit width, D, to the right, no lifting force other than hydrostatic, and no net moment. We affix a co-ordinate system to the body as shown in figure 1. In the following, primes denote dimensional variables and tildes dimensional perturbations of the dependent variables.

Far upstream, in the frame of reference of the body, the following conditions are assumed to exist:†

$$\left.\begin{aligned} \rho' &\to \rho_0(1-\beta y'), \\ u' &\to U, \\ v' &\to 0. \end{aligned}\right\} \tag{1}$$

We note that ρ_0 is the density of the fluid very far upstream at the level of the body and

$$\beta = \lim_{x'\to -\infty} \left|\frac{\partial \rho'}{\partial y'}\right| \Big/ \rho_0.$$

† For footnote see facing page.

We assume that the flow is laminar and steady, the fluid has uniform transport properties, and that β is so small that the Boussinesq approximation is valid.

We define the perturbations of the dependent variables as follows:

$$u' = U + \tilde{u}, \quad v' = \tilde{v}, \quad \rho' = \rho_0(1 - \beta y') + \tilde{\rho}, \qquad (2)$$
$$p' = -\rho_0 g(y' - \beta y'^2/2) + \tilde{p}.$$

We now only consider the flow field at distances from the body so large that the velocity perturbations are small compared with the undisturbed speed; the Oseen linearization of the convective derivative then applies. The governing equations under the above assumptions become

$$\rho_0 U \frac{\partial \tilde{u}}{\partial x'} + \frac{\partial \tilde{p}}{\partial x'} - \mu \nabla'^2 \tilde{u} = 0, \qquad (3a)$$

$$\rho_0 U \frac{\partial \tilde{v}}{\partial x'} + \frac{\partial \tilde{p}}{\partial y'} + \tilde{\rho} g - \mu \nabla'^2 \tilde{v} = 0, \qquad (3b)$$

$$\frac{\partial \tilde{u}}{\partial x'} + \frac{\partial \tilde{v}}{\partial y'} = 0, \qquad (3c)$$

$$U \frac{\partial \tilde{\rho}}{\partial x'} - \rho_0 \beta \tilde{v} = 0. \qquad (3d)$$

All perturbations are assumed to die out infinitely far from the body. The boundary conditions at the body may be replaced by a linearized momentum-integral equation as follows. If we integrate the Navier–Stokes equations over an appropriate volume containing the body and use the Boussinesq approximation and the assumption that all perturbations go to zero at infinity, we can show that

$$\int_{-\infty}^{+\infty} (\tilde{p} + \rho_0 \tilde{u}^2)\, dy' |_{x'=x'_+} - \int_{-\infty}^{+\infty} (\tilde{p} + \rho_0 \tilde{u}^2)\, dy' |_{x'=x'_-} = -D,$$

where x'_+ and x'_- are any two values of x' to the right and left of the body respectively. We now make the assumption, which we can verify *a posteriori*, that far upstream and downstream of the body the square of the horizontal velocity

† The assumption of an upstream density profile which is linear in y' over the entire range, $-\infty < y' < +\infty$, leads to large variation in ρ', violating the Boussinesq approximation, and to negative values of ρ' for $y' > \beta^{-1}$. Thus, the solution to the linear profile problem must be interpreted with care. Consider the following problem. Assume that upstream there exists a density profile which is linear in y' over the range $|y'| \leqslant Y$, and, as $|y'| \to \infty$, approaches, slowly, smoothly and monotonically, some positive values of ρ', say $\rho'_+ (\rho'_-)$ for $y' > Y (< -Y)$, with $0 < (\rho'_- - \rho'_+)/\rho'_- \ll 1$. Further we assume that, in the linear range, the fractional density change per unit height, β, is so small compared with one, that $\beta Y \ll 1$. The solution to the linear profile problem is expected to represent the flow field for this physically realistic profile in the range $|y'| \ll Y$. For $|y'| > Y$, the solution to the linear profile problem is of no physical significance. Since viscosity damps the perturbations in the vertical direction, most of the significant perturbations occur in the range $|y'| < Y$. Thus the solution obtained in this paper should represent the flow of a physically realistic density profile over that range of y' where most of the significant perturbations occur.

perturbation becomes small compared with the pressure perturbation. The momentum-integral relation then linearizes to

$$\int_{-\infty}^{+\infty} \tilde{p}\, dy'|_{x'=x'_+} - \int_{-\infty}^{+\infty} \tilde{p}\, dy'|_{x'=x'_-} = -D. \tag{4}$$

We now define a length, l, a Reynolds number based on this length, and non-dimensionalize as follows:

$$\left.\begin{aligned} &l = (U\nu/\beta g)^{\frac{1}{3}}, \quad Re_l = Ul/\nu = (U^4/\nu^2\beta g)^{\frac{1}{3}}, \\ &x' = xl, \quad y' = yl, \quad \tilde{u} = uD/\mu, \quad \tilde{v} = vD/\mu, \\ &\tilde{p} = pD/l, \quad \tilde{\rho} = \rho D/gl^2. \end{aligned}\right\} \tag{5}$$

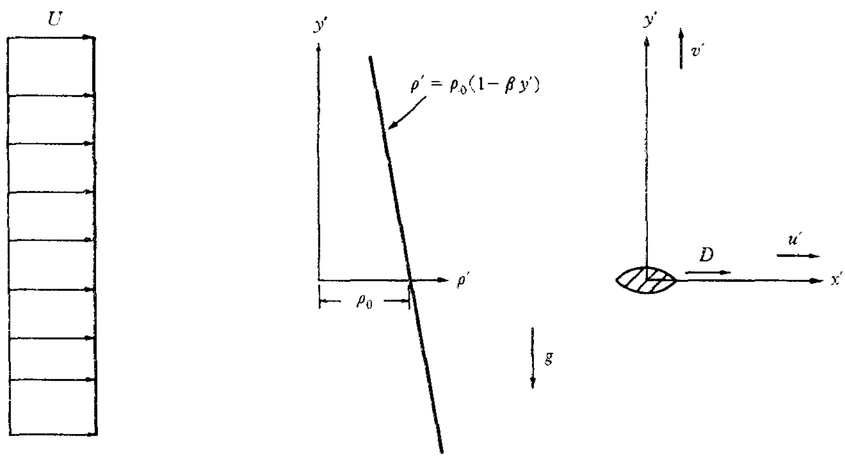

FIGURE 1. The geometry of the problem.

The governing equations now become

$$Re_l \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} - \nabla^2 u = -\delta(x)\delta(y), \tag{6a}$$

$$Re_l \frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} + \rho - \nabla^2 v = 0, \tag{6b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6c}$$

$$\frac{\partial \rho}{\partial x} - v = 0, \tag{6d}$$

and the conditions far from the body are

$$u, v, p, \rho \to 0 \quad \text{as} \quad |x^2 + y^2| \to \infty. \tag{7}$$

Condition (4) has been incorporated into the governing equations by placing the product of two Dirac delta functions on the right-hand side of (6a). We can verify that this condition is satisfied by first integrating (6a) with respect to y from minus to plus infinity; we then integrate with respect to x over an interval containing the origin and recover (4). This method of including the body effects by using the delta function was used by Childress (1964).

Before obtaining the solution of (6), we briefly investigate the significance of Re_l and examine a dispersion relation which we later show to be of some significance.

3. The significance of Re_l

We now consider a finite amplitude shear wave propagating horizontally to the left, with speed U, through a linearly stratified, non-diffusive viscous fluid. The disturbance is of the form

$$\left.\begin{aligned} u' &\equiv 0, \quad p' = -\rho_0 g(y' - \beta y'^2/2), \\ \rho' &= \rho_0(1-\beta y') + \tilde{\rho}(x', t'), \quad v' = \tilde{v}(x', t'). \end{aligned}\right\} \tag{8}$$

The only non-trivial Boussinesq equations are

$$\rho_0 \frac{\partial \tilde{v}}{\partial t'} + \tilde{\rho} g - \rho_0 \nu \frac{\partial^2 \tilde{v}}{\partial x'^2} = 0, \tag{9a}$$

$$\frac{\partial \tilde{\rho}}{\partial t'} - \rho_0 \beta \tilde{v} = 0. \tag{9b}$$

Differentiating (9a) with respect to time, we obtain

$$\frac{\partial^2 \tilde{v}}{\partial t'^2} + \tilde{v}\beta g - \nu \frac{\partial^3 \tilde{v}}{\partial t' \, \partial x'^2} = 0. \tag{10}$$

We assume a disturbance of the form

$$\tilde{v} = \tilde{v}_0 \exp[ik(x' + Ut')], \tag{11}$$

where U is a real positive number. For the Boussinesq approximation to be valid, we require $\beta \tilde{v}_0 \ll 1$. From (10), we obtain the dispersion relation

$$-k^2U^2 + \beta g + \nu i k^3 U = 0. \tag{12}$$

The first term in the above relation represents the inertial effect, I, the second the buoyant effect, B, and the third the viscous effect, V. We now form a dimensionless ratio of these effects as follows:

$$\frac{I^2}{V^2}\frac{I}{B} = U^4/\nu^2\beta g = Re_l^3. \tag{13}$$

Thus, the cube of the Reynolds number introduced in the preceding section corresponds to the square of the product of the internal Froude number with the internal Reynolds number of this shear wave.

Since (12) contains significant information affecting the downstream solution, as we will show in the next section, we consider it at length. First, we let

$$\zeta = -ik(U\nu/\beta g)^{\frac{1}{3}} = -ikl. \tag{14}$$

The disturbance (11) is then of the form

$$\tilde{v} = \tilde{v}_0 \exp[-\zeta l^{-1}(x' + Ut')], \tag{15}$$

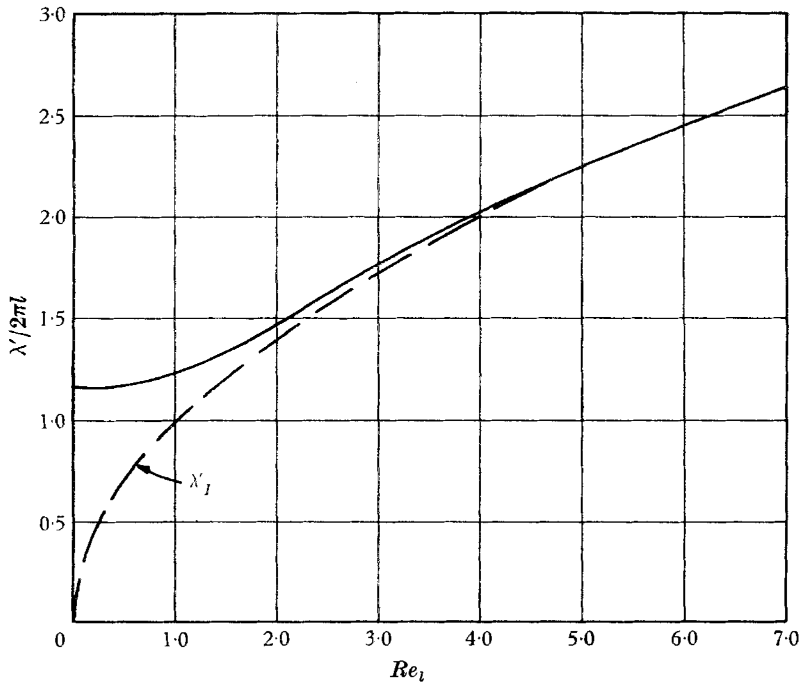

FIGURE 2. A plot of the wavelength of the viscous shear wave, λ', versus the Reynolds number. The dashed line is the inviscid wavelength. Note: $l = (U\nu/\beta g)^{\frac{1}{3}}$ and $Re_l = (U^4/\nu^2\beta g)^{\frac{1}{3}}$.

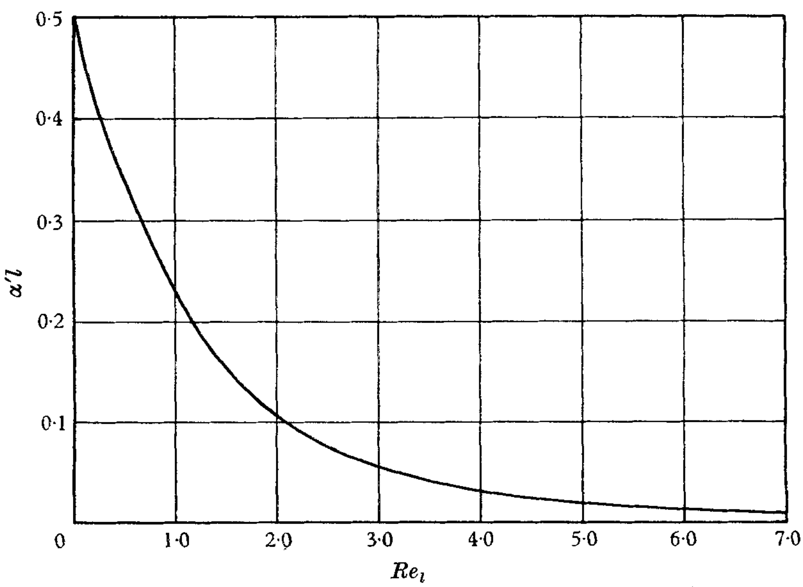

FIGURE 3. A plot of the attenuation constant of the viscous shear wave, α', versus the Reynolds number. Note: $l = (U\nu/\beta g)^{\frac{1}{3}}$ and $Re_l = (U^4/\nu^2\beta g)^{\frac{1}{3}}$.

and the dispersion relation is

$$\zeta^3 + Re_l \zeta^2 + 1 = 0. \tag{16}$$

This cubic has two complex conjugate roots with positive real parts, which we denote as $\zeta_R \pm i\zeta_I$, and a negative real root which is unimportant for the current discussion. Considering only the wave-like disturbance, we note that the dimensional wavelength, λ', is $2\pi l/\zeta_I$ and the dimensional attentuation constant, α', is ζ_R/l. The roots of (16) may be extracted analytically, and the results are plotted versus Re_l in figures 2 and 3. We can readily show that, for large Re_l,

$$\lambda' \sim 2\pi l (Re_l)^{\frac{1}{2}} \quad \text{and} \quad \alpha' \sim (2Re_l^2 l)^{-1} \sim 2\pi^2 (U\lambda'^2/\nu)^{-1}.$$

If the fluid were non-viscous, the inviscid wavelength, λ'_I, could be written as

$$\lambda'_I = 2\pi U/(\beta g)^{\frac{1}{2}} = 2\pi l U/(\beta g)^{\frac{1}{2}} l = 2\pi l (Re_l)^{\frac{1}{2}}.$$

This curve is also plotted in figure 2 to provide a measure of the effects of viscosity. We see that, at low speeds, viscosity increases the wavelength over its inviscid value. An observer moving with the wave sees an oncoming velocity, U, and the wave decaying in the downstream direction.

We now return to the solution of (6) and to a description of the entire flow field.

4. Solution of the governing equations

We assume the solution is sufficiently well behaved so that the Fourier transforms of the dependent variables exist; these transforms are defined by the following integrals which are assumed to exist as Cauchy principal values:

$$u(x,y) = (4\pi^2)^{-1} \iint_{-\infty}^{+\infty} \mathscr{U}(k_1,k_2) \exp[ik_1 x + ik_2 y]\, dk_1 dk_2, \tag{17a}$$

$$v(x,y) = (4\pi^2)^{-1} \iint_{-\infty}^{+\infty} \mathscr{V}(k_1,k_2) \exp[ik_1 x + ik_2 y]\, dk_1 dk_2, \tag{17b}$$

$$p(x,y) = (4\pi^2)^{-1} \iint_{-\infty}^{+\infty} \mathscr{P}(k_1,k_2) \exp[ik_1 x + ik_2 y]\, dk_1 dk_2, \tag{17c}$$

$$\rho(x,y) = (4\pi^2)^{-1} \iint_{-\infty}^{+\infty} \mathscr{R}(k_1,k_2) \exp[ik_1 x + ik_2 y]\, dk_1 dk_2. \tag{17d}$$

Substituting these expressions into (6), we obtain a system of four simultaneous, linear algebraic equations for the transforms which may be written as follows:

$$\begin{vmatrix} (Re_l ik_1 + k^2) & 0 & ik_1 & 0 \\ 0 & (Re_l ik_1 + k^2) & ik_2 & 1 \\ k_1 & k_2 & 0 & 0 \\ 0 & -1 & 0 & ik_1 \end{vmatrix} \begin{vmatrix} \mathscr{U} \\ \mathscr{V} \\ \mathscr{P} \\ \mathscr{R} \end{vmatrix} = \begin{vmatrix} -1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \tag{18}$$

where $k^2 = k_1^2 + k_2^2$. We solve this system of equations and obtain:

$$\mathscr{U}(k_1, k_2) = -\frac{k_2^2}{k_1^4 + iRe_l k_1^3 + 2k_2^2 k_1^2 - i(1 - Re_l k_2^2) k_1 + k_2^4}, \tag{19a}$$

$$\mathscr{V}(k_1, k_2) = +\frac{k_1 k_2}{k_1^4 + iRe_l k_1^3 + 2k_2^2 k_1^2 - i(1 - Re_l k_2^2) k_1 + k_2^4)}, \tag{19b}$$

$$\mathscr{P}(k_1, k_2) = +\frac{ik_1(Re_l ik_1 + k^2) + 1}{k_1^4 + iRe_l k_1^3 + 2k_2^2 k_1^2 - i(1 - Re_l k_2^2) k_1 + k_2^4}, \tag{19c}$$

$$\mathscr{R}(k_1, k_2) = -\frac{ik_2}{k_1^4 + iRe_l k_1^3 + 2k_2^2 k_1^2 - i(1 - Re_l k_2^2) k_1 + k_2^4}. \tag{19d}$$

The singularity of these transforms at $k_1 = k_2 = 0$ is due to the algebraic decay of the upstream solution and to the discontinuity of the 'pressure force' across the origin in the physical plane. However, as we shall see, these singularities do not affect the integrals of (17).

We observe from the above transforms and (17) that $u(x, y)$ and $p(x, y)$ are even functions of y, while $v(x, y)$ and $\rho(x, y)$ are odd functions of this variable. We now evaluate the first of (17) in detail.

Since $\mathscr{U}(k_1, k_2)$ is an even function of k_2, we may write

$$u(x, y) = (2\pi^2)^{-1} \int_0^\infty I_u(k_2, Re_l) \cos k_2 y \, dk_2, \tag{20}$$

where $$I_u(k_2, Re_l) = \int_{-\infty}^{+\infty} \frac{k_2^2 \exp[ik_1 x] \, dk_1}{k_1^4 + iRe_l k_1^3 + 2k_2^2 k_1^2 - i(1 - Re_l k_2^2) k_1 + k_2^4}. \tag{21}$$

If we consider k_1 to be a complex variable, then the path of integration in (21) is the real axis of the complex k_1 plane. We introduce a new complex variable, ζ, defined by

$$\zeta = -ik_1.$$

We may then write

$$I_u(k_2, Re_l) = i \int_{+i\infty}^{-i\infty} \frac{k_2^2 \exp[-\zeta x] \, d\zeta}{(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)(\zeta - \zeta_4)}, \tag{22}$$

where the $\zeta_i = \zeta_i(k_2, Re_l)$ are the four roots of

$$P(\zeta) = \zeta^4 + Re_l \zeta^3 - 2k_2^2 \zeta^2 + (1 - Re_l k_2^2) \zeta + k_2^4 = 0. \tag{23}$$

We shall discuss these roots in detail after considering the integration of (22). The path of integration of this integral is the imaginary axis of the ζ-plane. For each value of $k_2 \neq 0$, ζ_1 and ζ_2 have finite positive real parts, and ζ_3 and ζ_4 are real, negative, and finite. The situation is shown in figure 4.

To evaluate $I_u(k_2, Re_l)$ by the method of residues, we choose a closed contour lying in the ζ-plane consisting of the imaginary ζ-axis and a semi-circle centred at the origin, of infinite radius, and lying either in the half-plane $\zeta_R < 0$ or the half-plane $\zeta_R > 0$. If $x < 0$, we choose the former path, as the integral along this path vanishes; if $x > 0$, we choose the latter path for the same reason. Using the residue theorem, we obtain for $x > 0$

$$I_u(k_2, Re_l) = -2\pi \sum_{i=1}^{2} \frac{k_2^2 \exp[-\zeta_i x]}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{4} (\zeta_i - \zeta_j)}, \tag{24a}$$

and for $x < 0$
$$I_u(k_2, Re_l) = +2\pi \sum_{i=3}^{4} \frac{k_2^2 \exp[-\zeta_i x]}{\prod\limits_{\substack{j=1\\ j\neq i}}^{4} (\zeta_i - \zeta_j)}. \tag{24b}$$

These results hold for $k_2 \neq 0$. If $k_2 = 0$, ζ_1, ζ_2 and ζ_4 have finite real parts and $\zeta_3 = 0$. Thus
$$\left.\frac{I_u(k_2, Re_l)}{k_2^2}\right|_{k_2=0} = i\int_{+i\infty}^{-i\infty} \left.\frac{\exp[-\zeta x]d\zeta}{(\zeta-\zeta_1)(\zeta-\zeta_2)(\zeta-\zeta_4)\zeta}\right|_{k_2=0}.$$

The right-hand side of this expression can be shown to exist as a Cauchy principal value. Hence
$$I_u(0, Re_l) = 0.$$

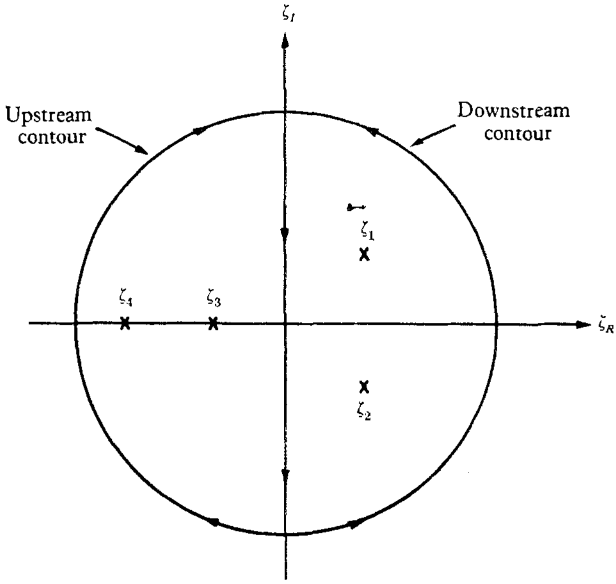

FIGURE 4. Contours of integration and the positions of the poles in the complex ζ-plane.

We can regard (24) as being valid for all values of k_2. Similar results hold for $v(x, y)$ and $\rho(x, y)$. The corresponding integral for the pressure is discontinuous, though finite at $k_2 = 0$; this is due to the jump in 'pressure force' across $x = 0$. However, a finite discontinuity at one point does not change the value of the integral, and so the singularities of the transforms at $k_1 = k_2 = 0$ are unimportant as far as our solution is concerned.

The general nature of the flow can be clarified by the following discussion which is based on an analysis of the roots of $P(\zeta)$.

(*a*) The roots whose residues contribute to the downstream solution, ζ_1 and ζ_2, are found to be complex conjugates, with positive real parts, for $0 \leqslant k_2 \leqslant k_2^*$, where k_2^* is a positive, monotonically decreasing function of Re_l. For $k_2 > k_2^*$, ζ_1 and ζ_2 are real, positive and unequal. We can then write for $x > 0$

$$u(x,y) = -\pi^{-1}\int_0^{k_2^*} \frac{k_2^2 \exp(-\zeta_{1R}x)\sin(\zeta_{1I}x+\theta)\cos k_2 y}{\zeta_{1I}\sqrt{\{(\zeta_{1R}-\zeta_3)^2+\zeta_{1I}^2\}}\cdot\sqrt{\{(\zeta_{1R}-\zeta_4)^2+\zeta_{1I}^2\}}}dk_2$$
$$+\pi^{-1}\int_{k_2^*}^{\infty}\left[\frac{k_2^2\exp(-\zeta_1 x)\cos k_2 y}{(\zeta_1-\zeta_2)(\zeta_1-\zeta_3)(\zeta_1-\zeta_4)} + \frac{k_2^2\exp(-\zeta_2 x)\cos k_2 y}{(\zeta_2-\zeta_1)(\zeta_2-\zeta_3)(\zeta_2-\zeta_4)}\right]dk_2, \tag{25a}$$

where $$\zeta_1 = \zeta_{1R} + i\zeta_{1I} \quad (k_2 \leqslant k_2^*),$$

and $$\theta = \tan^{-1}\left(\frac{\zeta_{1I}}{\zeta_{IR} - \zeta_3}\right) + \tan^{-1}\left(\frac{\zeta_{1I}}{\zeta_{IR} - \zeta_4}\right),$$

and for $x < 0$

$$u(x,y) = -\pi^{-1}\int_0^\infty \left[\frac{k_2^2 \exp(-\zeta_3 x)\cos k_2 y}{(\zeta_3-\zeta_1)(\zeta_3-\zeta_2)(\zeta_3-\zeta_4)} + \frac{k_2^2 \exp(-\zeta_4 x)\cos k_2 y}{(\zeta_4-\zeta_1)(\zeta_4-\zeta_2)(\zeta_4-\zeta_3)}\right] dk_2. \tag{25b}$$

Equation (25*a*), which describes the downstream behaviour, is composed of two integrals. The first contains x in the exponent of $\exp(-\zeta_{1R}x)$ and $\sin(\zeta_{1I}x+\theta)$. We might expect this integral to exhibit an oscillatory behaviour with x, and we call this integral the wave-like component. In the next section, we show that this integral does have an oscillatory behaviour. The second integral of (25*a*) contains x in the exponents of $\exp(-\zeta_1 x)$ and $\exp(-\zeta_2 x)$. We expect this integral to decay exponentially with x. The upstream solution, as we show in the next section, decays algebraically with x.

(*b*) When k_2 increases from k_2^*, ζ_1 increases monotonically and ζ_2 first decreases to a minimum value of $4/Re_l^2$ at $k_2^2 = 2/Re_l + 16/Re_l^4$ and then increases monotonically. Since $4/Re_l^2 > \zeta_{1R}(0, Re_l)$ for all Re_l, the wave-like component of the solution predominates far downstream.

(*c*) As k_2 increases from zero to k_2^*, ζ_{1I} decreases monotonically to zero. The wave-like component is, thus, a superposition of waves whose wave-numbers in the x-direction are less than $\zeta_{1I}(0, Re_l)$. ζ_{1R} increases monotonically with k_2 in this range. Thus, far downstream, only the shortest waves, with wave-numbers $\lesssim \zeta_{1I}(0, Re_l)$, persist. For small k_2, ζ_1 is approximately the complex root of (16), since $P(\zeta)$ reduces to this relation when k_2 goes to zero. Thus, the information given in figures 2 and 3 applies, approximately, far downstream of the body.

(*d*) For a fixed value of k_2, as Re_l increases, ζ_1 and ζ_2 decrease. Thus, because of the exponential decay of the integrand of (25*a*), as Re_l increases, the downstream perturbation increases.

(*e*) The roots whose residues contribute to the upstream solution, ζ_3 and ζ_4, are real, negative, and unequal, monotonically decreasing functions of k_2. ζ_3 goes to zero, for all Re_l, with k_2 as $-k_2^4$. Thus, the upstream solution has an algebraically decaying component, as we shall show in the next section, due to the residue of ζ_3. At $k_2 = 0$, $\zeta_4(0, Re_l) \leqslant -1$; the maximum occurs when $Re_l = 0$. The upstream component due to the residue of ζ_4 decays at least as fast as $\exp(-|x|)$ and is negligible compared with the other component.

(*f*) For a fixed non-zero value of k_2, as Re_l increases, ζ_3 and ζ_4 become more negative; this shows the upstream perturbations decrease with increasing Re_l.

Thus, the downstream behaviour is wave-like with the wavelengths and attentuation constants given approximately in figures 2 and 3. With an increase in speed, these waves increase in wavelength and decay more slowly. An asymptotic formula for the vertical velocity is derived in the next section.

Numerical integration of (25*b*) shows a system of jets upstream of the body. These jets become wider and decay algebraically in the upstream direction. As the Reynolds number increases, the upstream jets become weaker and wider.

Profiles of the horizontal velocity perturbations are shown in figures 5, 6, and 7 for a range of Reynolds numbers; u_{Ax} is the horizontal velocity perturbation at $y = 0$ for the specified value of x. As we note in the next section, the profile in figure 5 for $x = -1000$ is the asymptotic profile. From this figure we see that th jet centred near $y = 20$ is more than half the strength of that centred about $y = 0$. The horizontal velocity perturbations at $y = 0$ are plotted versus distance

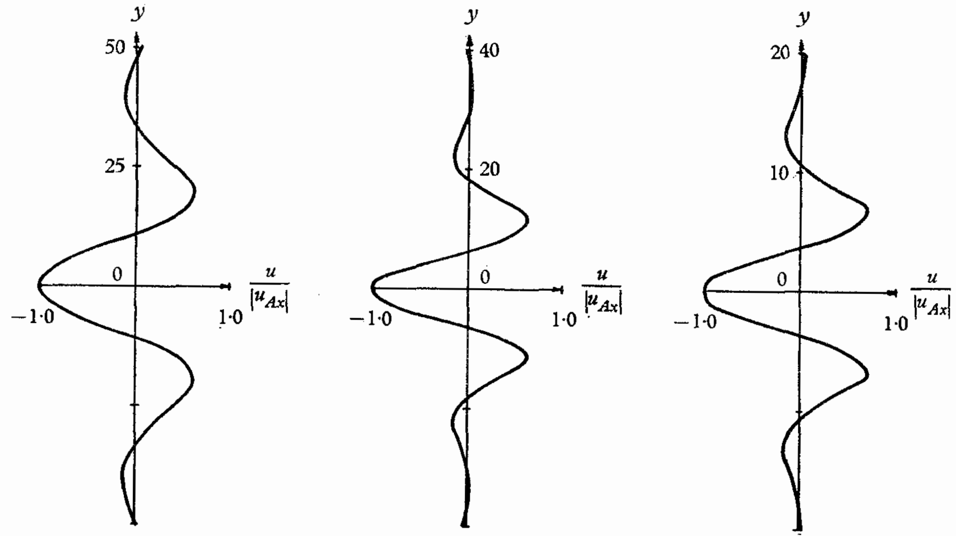

FIGURE 5. Profiles of the horizontal velocity perturbation for a fixed Reynolds number at three stations upstream. These stations are, from left to right, $x = -1000$, -100, -10. Also, $u_{Ax} = u(x, 0)$, the horizontal velocity perturbation along the x-axis. For this figure, we have $Re_l = 1{\cdot}0$ and $u(-1000, 0) = -5{\cdot}45 \times 10^{-4}$, $u(-100, 0) = -3{\cdot}03 \times 10^{-3}$, $u(-10, 0) = -1{\cdot}65 \times 10^{-2}$.

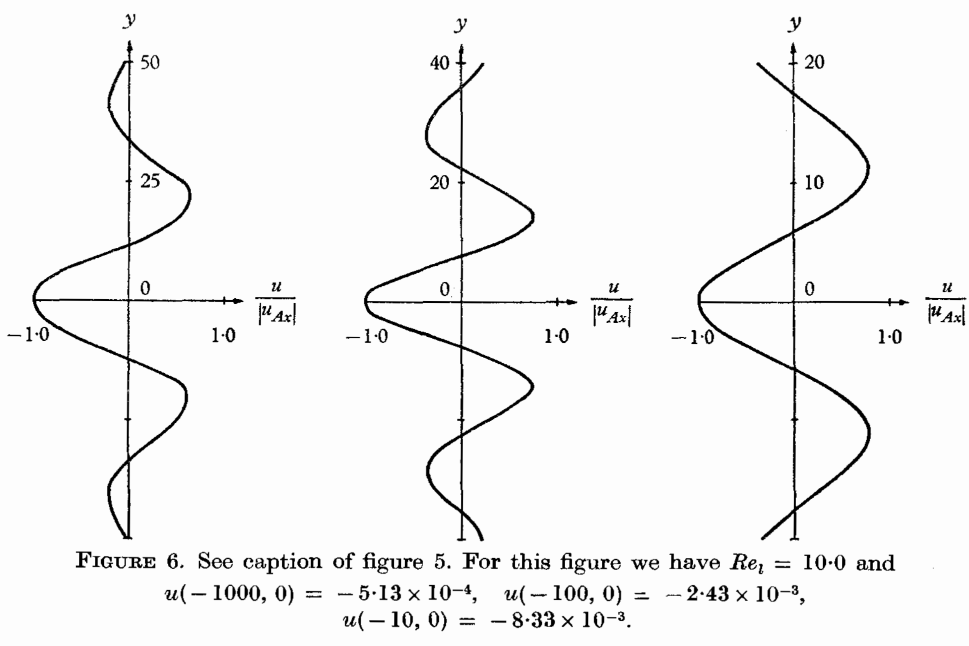

FIGURE 6. See caption of figure 5. For this figure we have $Re_l = 10{\cdot}0$ and $u(-1000, 0) = -5{\cdot}13 \times 10^{-4}$, $u(-100, 0) = -2{\cdot}43 \times 10^{-3}$, $u(-10, 0) = -8{\cdot}33 \times 10^{-3}$.

upstream in figure 8, for a range of Reynolds numbers. The dimensional perturbation, $\tilde{u}(x', y')$, is related to the dimensionless values shown in these figures through equation (5). In §5, we obtain formulae for the perturbations which hold asymptotically far upstream; these are shown to be identical with Long's similarity solution and are related directly to the drag.

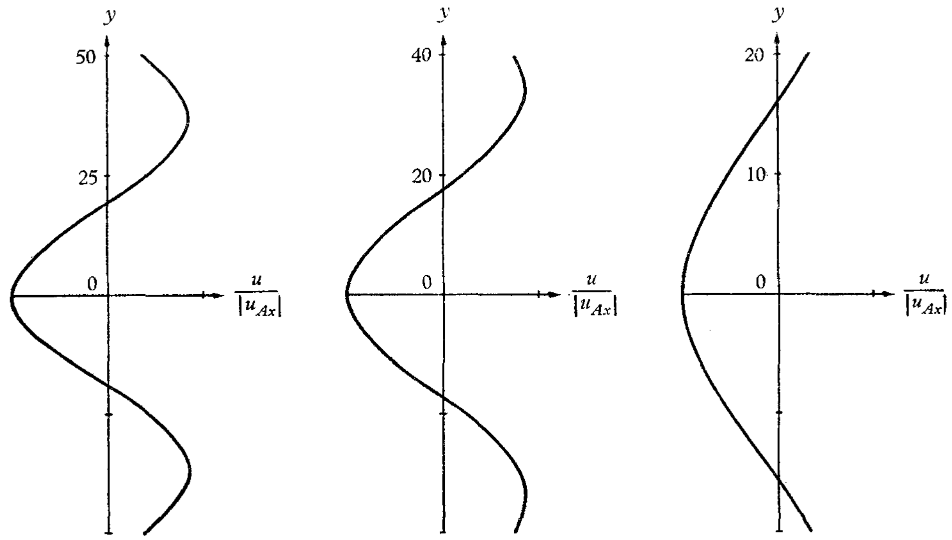

FIGURE 7. See caption of figure 5. For this figure, we have $Re_l = 100{\cdot}0$ and $u(-1000, 0) = -2{\cdot}62 \times 10^{-4}$, $u(-100, 0) = -5{\cdot}17 \times 10^{-4}$, $u(-10, 0) = -9{\cdot}50 \times 10^{-4}$. By comparing figures 5, 6 and 7, we see that, as Re_l increases, the jets at a fixed x become wider.

5. The asymptotic formulae

We first consider the upstream solution. It is clear from the exponential decay of the integrand of (25*b*) that for very large values of $|x|$ only the first term contributes significantly to the integral. Furthermore, since ζ_3 increases in magnitude monotonically with k_2, only the contributions for small k_2 are significant. For small values of k_2, we see that

$$\zeta_3 = -k_2^4 - Re_l k_2^6 + O(k_2^8),$$

since

$$P(-k_2^4 - Re_l k_2^6) = O(k_2^8).$$

Approximating the other roots for small k_2, and considering values of $|x|$ so large and values of ϵ so small that

$$\epsilon^4 |x| \gg 1$$

and

$$Re_l \epsilon^6 |x| \ll 1,$$

we are able to write

$$u(x, y) = -\pi^{-1} \int_0^{\epsilon} k_2^2 \cos k_2 y \exp(-k_2^4 x)(1 + O(\epsilon))\, dk_2 + O(\epsilon^4). \tag{26}$$

The requirement on $|x|$, which follows from the above two inequalities, is that

$$|x| \gg Re_l^2. \tag{27}$$

This leads to the solution

$$\tilde{u}(x',y') = -\pi^{-1}(D/\mu)(U\nu/\beta g)^{\frac{1}{4}}|x'|^{-\frac{3}{4}}\int_0^\infty s^2\exp(-s^4)\cos\eta s\,ds, \tag{28a}$$

$$\tilde{v}(x',y') = +\pi^{-1}(D/\mu)(U\nu/\beta g)^{\frac{1}{2}}|x'|^{-\frac{3}{2}}\int_0^\infty s^5\exp(-s^4)\sin\eta s\,ds, \tag{28b}$$

$$\tilde{p}(x',y') = +\pi^{-1}D(\beta g/U\nu)^{\frac{1}{4}}|x'|^{-\frac{1}{4}}\int_0^\infty \exp(-s^4)\cos\eta s\,ds, \tag{28c}$$

$$\tilde{\rho}(x',y') = +\pi^{-1}(D/g)(\beta g/U\nu)^{\frac{1}{2}}|x'|^{-\frac{1}{2}}\int_0^\infty s\exp(-s^4)\sin\eta s\,ds, \tag{28d}$$

$$\tilde{\psi}(x',y') = +\pi^{-1}(D/\mu)(U\nu/\beta g)^{\frac{1}{2}}|x'|^{-\frac{1}{2}}\int_0^\infty s\exp(-s^4)\sin\eta s\,ds, \tag{28e}$$

where
$$\tilde{u} = -\frac{\partial\tilde{\psi}}{\partial y'},\quad \tilde{v} = \frac{\partial\tilde{\psi}}{\partial x'},$$

and
$$\eta = (\beta g/U\nu)^{\frac{1}{4}}y'/|x'|^{\frac{1}{4}}.$$

We can readily show that the above solution is the integral form of Long's (1959) similarity solution with the perturbations related directly to the drag. Long obtained his solution by neglecting inertia and making the boundary-layer approximation; (28) shows that this procedure is valid far upstream of the body. For $Re_l = 1{\cdot}0$ and $|x| > 100$, the values of $u(x,0)$ given by (28a) agree with the results of the numerical integration of (25b), and so the curve for $Re_l = 1{\cdot}0$ in figure 8 corresponds to the asymptotic values of $u(x,0)$.

We now develop a formula, valid at high Re_l, for the vertical velocity far downstream of the body. The integral for the vertical velocity, analogous with (25a), has a wave-like component and a purely exponentially decaying component. Far downstream, the latter is negligible compared with the former, and we may write

$$v(x,y) \cong (2\pi)^{-1}\frac{\partial}{\partial y}\int_0^{k_2^*}\frac{|\zeta_1|\exp(-\zeta_{1R}x)\,dk_2}{\zeta_{1I}|\zeta_1-\zeta_3||\zeta_1-\zeta_4|}$$
$$\times[\sin(\zeta_{1I}x+k_2y+\theta)+\sin(\zeta_{1I}x-k_2y+\theta)],$$

where
$$\theta = \tan^{-1}\left(\frac{\zeta_{1I}}{\zeta_{1R}-\zeta_3}\right)+\tan^{-1}\left(\frac{\zeta_{1I}}{\zeta_{1R}-\zeta_4}\right)-\tan^{-1}\left(\frac{\zeta_{1I}}{\zeta_{1R}}\right).$$

We note that $v(x,y)$ is an odd function of y, and we now consider only $y \geqslant 0$. For large values of x, the arguments of the sinusoids vary rapidly with k_2. However, the first sinusoid has one value of k_2 where the argument is stationary. To obtain a simple asymptotic result, we make the following restrictions:

$$Re_l \gg 1 \quad\text{and}\quad (y'/x')^2 \ll U^3/\nu\beta g x' \ll 1.$$

Using (23), we obtain power series expansions for the various roots in powers of k_2; for example,

$$\zeta_{1I} \approx \frac{1}{\sqrt{Re_l}}-\frac{(Re_l)^{\frac{1}{2}}}{2}k_2^2+O(k_2^4).$$

We may then approximate the terms in the integrand of the above expression for $v(x,y)$ and solve for the value of k_2 which causes the argument of the sinusoid to be stationary, i.e. its derivative with respect to k_2, to vanish. A straightforward, but tedious, application of Kelvin's principle of stationary phase, as described in Sneddon (1951), then gives as a first approximation:

$$\tilde{v}(x',y') = \frac{D(\beta g)^{\frac{1}{4}}}{\rho_0 U^{\frac{3}{2}}} \frac{y'}{(2\pi x'^3)^{\frac{1}{2}}} \exp(-\beta g \nu x'/U^3) \cos\left[\frac{(\beta g)^{\frac{1}{2}}}{U}\left(x' + \frac{y'^2}{2x'}\right) - \frac{\pi}{4}\right]. \quad (29)$$

The attentuation constant and the wave-number are as given in figures 2 and 3. The numerical integration of (25a), using Simpson's rule, gives results which agree well with those of (29) in the range where the latter is expected to apply.

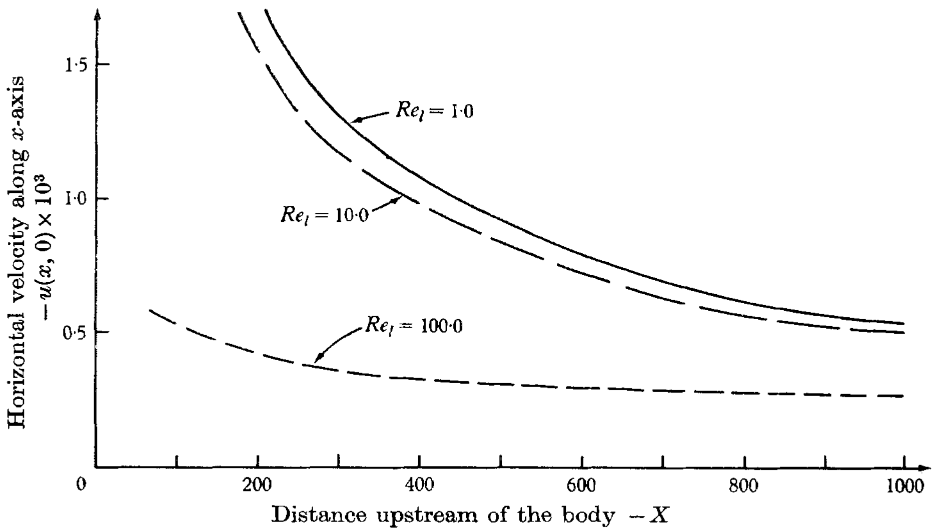

FIGURE 8. The horizontal velocity perturbation along the x-axis upstream of the body for various values of Re_l.

If we attempt to confirm the results of (28) experimentally, we must resolve two difficulties. First, the location of the effective force is unknown with respect to the body geometry. To overcome this, we may force agreement between observed and theoretical values of the velocity at one point very far upstream and so determine the origin of the far field co-ordinate system. Also, the effects of body shape are undetermined. However, if we make observations far from the body, in terms of its own dimensions, these effects should be small. Of course, our linear theory should apply only in the region so far from the body that

$$|\tilde{u}| \ll U,$$

We can use (28a) to determine the upstream distance where the Oseen approximation becomes valid by requiring that

$$\left|\frac{\tilde{u}(x',y')}{U}\right| \leqslant \left|\frac{\tilde{u}(x',0)}{U}\right| = 0{\cdot}975 D\rho_0^{-1}(\beta g)^{-\frac{1}{4}}(U\nu|x'|)^{-\frac{3}{4}} \ll 1$$

or

$$|x'| \gg (0{\cdot}975 D/\rho_0)^{\frac{4}{3}}(U\nu)^{-1}(\beta g)^{-\frac{1}{3}}.$$

6. Summary and conclusions

Upstream of the body a system of alternating jets exists. These jets decay algebraically and become wider in the upstream direction. For a fixed value of the drag and at a fixed dimensional distance on the x'-axis upstream of the body, as the speed of the body increases, the ratio of the velocity perturbation to the speed of the body decreases in magnitude; the width of the jets at a fixed value of x' increases with speed. The asymptotic solution given in (28) is valid far upstream of the body. For low speeds ($Re_l \leqslant 1 \cdot 0$), this solution becomes valid close to the body.

Downstream of the body, a pattern of waves exists; i.e. the streamlines oscillate about their equilibrium heights. Far downstream, the information contained in figures 2 and 3 applies. These figures show that, as the speed of the body increases, the lee waves increase in wavelength and attenuate more slowly.

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